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Current propagators and spectral sum rules for large and small momentum

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Abstract. Expanding the current propagators in the small and large momentum limit, in both the free field and spectral representations, is shown to lead to Weinberg's first sum rule for SU(3), or U(3), and either the Oakes–Sakurai or Das–Mathur–Okubo sum rules for cases of current or mass mixing, respectively. Resulting mass and coupling constant sum rules are listed for various choices of symmetry breaking and predictions of mixing angle and isovector masses for spin 1 and spin 0 mesons given. The method also allows systematic introduction of higher order symmetry breaking through the moment spectral integrals.

1. Introduction

Postulating nonets of vector and axial vector currents, $V_a^\mu(x)$ and $A_a^\mu(x)$, with the local commutation relationships (μ Lorentz indices, a, b unitary indices),

$$\begin{aligned}
[V_a^0(x), V_b^\mu(y)]\delta(x^0-y^0) &= \mathrm{i}f_{abc}V_c^\mu(y)\delta(x-y)+\mathrm{i}\delta_{ab}s\partial^\mu\delta(\boldsymbol{x}-\boldsymbol{y})\\
[V_a^0(x), A_b^\mu(y)]\delta(x^0-y^0) &= \mathrm{i}f_{abc}A_c^\mu(y)\delta(x-y)\\
[A_a^0(x), V_b^\mu(y)]\delta(x^0-y^0) &= -\mathrm{i}f_{abc}A_c^\mu(y)\delta(x-y)\\
[A_a^0(x), A_b^\mu(y)]\delta(x^0-y^0) &= \mathrm{i}f_{abc}V_c^\mu(y)\delta(x-y)+\mathrm{i}\delta_{ab}v\partial^\mu\delta(\boldsymbol{x}-\boldsymbol{y})
\end{aligned} \tag{1.1}$$

with s and v symmetric c numbers and f_{abc} the structure constants of SU(3) (or U(3)), insures that the twice integrated commutators satisfy the charge algebra originally postulated by Gell-Mann (1962, 1964). Furthermore, in the limit of conserved vector and axial vector currents, Weinberg (1967) showed $s = v$, which, using the now familiar spectral representation for the current propagators

$$\begin{aligned}
\int \mathrm{d}^4x\, \mathrm{e}^{-\mathrm{i}q\cdot x}\langle 0|TJ_a^\mu(x), J_b^\nu(0)|0\rangle
&= -\mathrm{i}\int \mathrm{d}m^2\rho_{ab}^{(1)}(m^2)\frac{g^{\mu\nu}-m^{-2}q^\mu q^\nu}{q^2-m^2}+\mathrm{i}q^\mu q^\nu\int \mathrm{d}m^2\rho_{ab}^{(0)}(m^2)\frac{1}{q^2-m^2}\\
&\quad +\text{Schwinger terms},
\end{aligned} \tag{1.2}$$

with $J_a^\mu(x)$ designating $V_a^\mu(x)$ or $A_a^\mu(x)$ and $\rho^{(1)}(m^2)$, $\rho^{(0)}(m^2)$ the spin 1 and spin 0 spectral functions, leads to the chiral sum rule

$$\delta_{ab}v = \int \mathrm{d}m^2\left(\frac{\rho_{ab}^{(1^+)}(m^2)}{m^2}+\rho_{ab}^{(0^-)}(m^2)\right) = \int \mathrm{d}m^2\left(\frac{\rho_{ab}^{(1^-)}(m^2)}{m^2}+\rho_{ab}^{(0^+)}(m^2)\right) = \delta_{ab}s, \tag{1.3}$$

where 1^+, 0^-, 1^-, and 0^+ designate axial vector, pseudoscalar, vector and scalar meson unitary indices respectively. It is convenient to define the spin 1 and spin 0 current propagators independently in the spectral representation, dropping the signature $\pm$, since equations are applicable to both states,

$$\begin{aligned}\Delta_{ab}^{\mu\nu}(q) &= \int \mathrm{d}m^2 \rho_{ab}^{(1)}(m^2)\frac{g^{\mu\nu}-m^{-2}q^\mu q^\nu}{q^2-m^2}\\ \Delta_{ab}(q) &= \int \mathrm{d}m^2 \rho_{ab}^{(0)}(m^2)\frac{m^4}{q^2-m^2}.\end{aligned} \tag{1.4}$$

Schnitzer and Weinberg (1968) showed, using meson dominance in the broad sense (the momentum dependence of an n point function of currents arises from its meson poles almost entirely) and smoothness, that the inverse current propagators must be at most quadratic in four-momentum, suppressing also for the moment the unitary indices

$$\Delta^{\mu\nu}(q)^{-1} \sim (\alpha+\beta q^2)g^{\mu\nu}+\gamma q^\mu q^\nu, \tag{1.5}$$

with α, β, γ functions of the unitary indices. Obviously, this gives

$$(\alpha+\beta q^2)\Delta^{\mu\nu}(q) \sim g^{\mu\nu}-\gamma\{\alpha+(\beta+\gamma)q^2\}^{-1}q^\mu q^\nu. \tag{1.6}$$

From the first of equations (1.4), we obtain a constraint

$$q_\mu\Delta^{\mu\nu}(q) = -q^\nu \int \mathrm{d}m^2 \frac{\rho^{(1)}(m^2)}{m^2} \tag{1.7}$$

requiring $q_\mu\Delta^{\mu\nu}(q)$ proportional to q^ν so that, in equation (1.6),

$$q_\mu\Delta^{\mu\nu}(q) \sim \{\alpha+(\beta+\gamma)q^2\}^{-1}q^\nu \tag{1.8}$$

and therefore $\beta = -\gamma$. One can, therefore, associate $\Delta^{\mu\nu}(q)$ with free-field propagators of masses α/β and coupling constants β^{-1} and cast the spin 1 and spin 0 propagators in the form,

$$\begin{aligned}\Delta^{\mu\nu}(q) &= \frac{g^{\mu\nu}+\beta\alpha^{-1}q^\mu q^\nu}{\alpha+\beta q^2}\\ \Delta(q) &= \frac{1}{\epsilon+\omega q^2},\end{aligned} \tag{1.9}$$

with ϵ and ω playing roles similar to α and β. Meson dominance, in addition to prescribing a type of free-field behaviour for the propagators, implies single particle saturation of the spectral functions which, in the absence of mixing of unitary states, implies,

$$\begin{aligned}\rho_{ab}^{(1)}(m^2) &= \delta(m^2-m_a^2)g_a^2\delta_{ab}\\ \rho_{ab}^{(0)}(m^2) &= \delta(m^2-m_a^2)f_a^2\delta_{ab},\end{aligned} \tag{1.10}$$

where g_a and f_a define the usual vacuum to single particle current matrix elements for the various meson states. To obtain further results in this scheme, it is necessary to make additional assumptions about the symmetry of the current propagators. In the following section we examine the large and small momentum expansions of the propagators and make the ansatz of current mixing and mass mixing for the spin 1 and spin 0 propagators and admit mixing of the octet and singlet states. The appendix gives a short discussion of mass and current mixing and types of symmetry breakings of interest.

2. Sum rules and symmetry breaking

The particular choices for symmetry breaking are made in the parameters α, β, ϵ and ω. The choice of α or ϵ broken and β or ω symmetric is called mass mixing, while the opposite, α or ϵ symmetric and β or ω broken, is called current mixing (Coleman and Schnitzer 1964, Kroll *et al* 1967). The particular choices of breaking are left unspecified at this point, and the large and small q^2 limits are first taken to obtain two types of sum rules involving these parameters. Taking the small and large q^2 limits in equations (1.9) and equations (1.4), and equating yields, again suppressing the unitary indices on $\Delta^{\mu\nu}$, Δ, α, β, ϵ, ω, $\rho^{(1)}$, and $\rho^{(0)}$,

$$
\begin{aligned}
\lim_{q^2\to 0} \Delta^{\mu\nu}(q) &\simeq g^{\mu\nu}(\alpha^{-1} - \beta\alpha^{-2}q^2 + \ldots) + \ldots \\
&= -g^{\mu\nu}\int \mathrm{d}m^2 \frac{\rho^{(1)}(m^2)}{m^2}(1 + m^{-2}q^2 + \ldots) + \ldots \\
\lim_{q^2\to 0} \Delta(q) &\simeq (\epsilon^{-1} - \omega\epsilon^{-2}q^2 + \ldots) \\
&= -\int \mathrm{d}m^2 \frac{\rho^{(0)}(m^2)}{m^2}(1 + m^{-2}q^2 + \ldots)
\end{aligned}
\tag{2.1}
$$

and,

$$
\begin{aligned}
\lim_{q^2\to\infty} \Delta^{\mu\nu}(q) &\simeq \frac{q^\mu q^\nu}{q^2\alpha}(1 - \alpha\beta^{-1}q^{-2} + \ldots) + \ldots \\
&= -\frac{q^\mu q^\nu}{q^2}\int \mathrm{d}m^2 \frac{\rho^{(1)}(m^2)}{m^2}(1 + m^2 q^{-2} + \ldots) + \ldots \\
\lim_{q^2\to\infty} \Delta(q) &\simeq \frac{1}{q^2\omega}(1 - \epsilon\omega^{-1}q^{-2} + \ldots) \\
&= \frac{1}{q^2}\int \mathrm{d}m^2 \rho^{(0)}(m^2)(1 + m^2 q^{-2} + \ldots).
\end{aligned}
\tag{2.2}
$$

Using the constraint implied by the chiral sum rule, equation (1.3), and equating coefficients of powers of q^2 gives for the $q^2 \to 0$ case

$$
\begin{aligned}
&\int \mathrm{d}m^2 \frac{\rho^{(1)}(m^2)}{m^2} = -\alpha^{-1} \\
&\int \mathrm{d}m^2 \frac{\rho^{(1)}(m^2)}{m^4} = \beta\alpha^{-2} \\
&\int \mathrm{d}m^2 \rho^{(0)}(m^2) = v + \alpha^{-1} = s + \alpha^{-1} \\
&\int \mathrm{d}m^2 \frac{\rho^{(0)}(m^2)}{m^2} = -\epsilon^{-1},
\end{aligned}
\tag{2.3}
$$

and correspondingly for the $q^2 \to \infty$ case

$$\begin{aligned}
&\int \mathrm{d}m^2 \frac{\rho^{(1)}(m^2)}{m^2} = -\alpha^{-1} \\
&\int \mathrm{d}m^2 \rho^{(1)}(m^2) = \beta^{-1} \\
&\int \mathrm{d}m^2 \rho^{(0)}(m^2) = \omega^{-1} = v+\alpha^{-1} = s+\alpha^{-1} \\
&\int \mathrm{d}m^2 \rho^{(0)}(m^2) m^2 = -\epsilon\omega^{-2}.
\end{aligned} \tag{2.4}$$

The first set of sum rules for the spin 1 mesons, corresponding to $q^2 \to 0$ are called the Oakes–Sakurai (OS) sum rules (Oakes and Sakurai 1967), while the second set, for $q^2 \to \infty$ are the Das–Mathur–Okubo (DMO) sum rules (Das *et al* 1967). The OS set will lead to $(\text{mass})^{-2}$ sum rules and the DMO set to $(\text{mass})^2$ sum rules for spin 1 and spin 0 mesons. Consistent with the requirement of current mixing for spin 1 mesons and mass mixing for spin 0 mesons, we redefine the various constants appearing on the right hand sides of equations (2.3) and (2.4) and explicitly exhibit the dependences on the unitary indices. We write equations (2.3) and (2.4) in the final form, restoring specific use of the unitary indices,

$$\begin{aligned}
&\int \mathrm{d}m^2 \frac{\rho^{(1)}_{ab}(m^2)}{m^2} = s^{(1)}\delta_{ab} \\
&\int \mathrm{d}m^2 \frac{\rho^{(1)}_{ab}(m^2)}{m^4} = t^{(1)}_{ab} \\
&\int \mathrm{d}m^2 \rho^{(0)}_{ab}(m^2) = s^{(0)}\delta_{ab} \\
&\int \mathrm{d}m^2 \frac{\rho^{(0)}_{ab}(m^2)}{m^2} = t^{(0)}_{ab},
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
&\int \mathrm{d}m^2 \frac{\rho^{(1)}_{ab}(m^2)}{m^2} = s^{(1)}\delta_{ab} \\
&\int \mathrm{d}m^2 \rho^{(1)}_{ab}(m^2) = u^{(1)}_{ab} \\
&\int \mathrm{d}m^2 \rho^{(0)}_{ab}(m^2) = s^{(0)}\delta_{ab} \\
&\int \mathrm{d}m^2 \rho^{(0)}_{ab}(m^2) m^2 = u^{(0)}_{ab},
\end{aligned} \tag{2.6}$$

where $-\alpha^{-1} = s^{(1)}\delta_{ab}$, $\beta\alpha^{-2} = t^{(1)}_{ab}$, $(v+\alpha^{-1}) = (s+\alpha^{-1}) = \omega^{-1} = s^{(0)}\delta_{ab}$, $-\epsilon^{-1} = t^{(0)}_{ab}$, $\beta^{-1} = u^{(1)}_{ab}$, $\epsilon\omega^{-2} = u^{(0)}_{ab}$ and $s^{(1)}+s^{(0)} = v = s$. The breaking is made in the terms t and u and is predicted on a general isospin conserving mechanism of the form ($i = 0, 1$, see appendix),

$$t^{(i)}_{ab} = u^{(i)}_{ab} = A\delta_{ab} + Bd_{8ab} + C\delta_{a8}\delta_{b8} + D\delta_{a0}\delta_{b0} + E(\delta_{a0}\delta_{b8} + \delta_{a8}\delta_{b0}) \tag{2.7}$$

with d_{8ab} the symmetrical structure constants. The choice $D = C/8$, $E = -\sqrt{2}C/5$ represents $8 \oplus 27$ breaking (Chan *et al* 1969), while $C = D = E = 0$ gives octet breaking (Oakes 1968, Wienke and Deshpande 1969). The choice $C = E = 0$ gives octet–singlet splitting in U(3) (Schwinger 1964) and $C = 0$ represents octet splitting with mixing in U(3) (Das *et al* 1967). Other models may be incorporated by judicious choice of the parameters A, B, C, D and E. Results for the models indicated are given in the next section.

The set of spin 1 sum rules has already been obtained using slightly different techniques (Oakes and Sakurai 1967, Das *et al* 1967) from the algebraic expansion in q^2 employed herein. In addition to providing a simple and consistent derivation of the OS and DMO sum rules for both spin 1 and spin 0 mesons, this approach allows for systematic introduction of higher order symmetry breaking through the various moment (in m^2) spectral integrals appearing as coefficients of higher orders of q^2. This is accomplished simply by making α, β, ϵ and ω functions of q^2, or related quantities, consistent with other models which predict the behaviour of the propagators. Obviously, the approach described above underscores a perturbative expansion of the symmetry breaking.

3. Results

It will be convenient to first enumerate the coupling constants in the scheme of mixing of 0 and 8 unitary meson components. The spectral functions for the spin 1 mesons are written in U(3),

$$\begin{aligned}
\rho^{(1)}_{11}(m^2) &= \rho^{(1)}_{22}(m^2) = \rho^{(1)}_{33}(m^2) = m_3^4 f_3^{-2}\delta(m^2-m_3^2)\\
\rho^{(1)}_{44}(m^2) &= \rho^{(1)}_{55}(m^2) = \rho^{(1)}_{66}(m^2) = \rho^{(1)}_{77}(m^2) = m_4^4 f_4^{-2}\delta(m^2-m_4^2)\\
\rho^{(1)}_{88}(m^2) &= \tfrac{3}{4} f_Y^{-2}\{m_0^4\sin^2\xi_Y\delta(m^2-m_0^2)+m_8^4\cos^2\xi_Y\delta(m^2-m_8^2)\}\\
\rho^{(1)}_{00}(m^2) &= \tfrac{3}{2} f_B^{-2}\{m_0^4\cos^2\xi_B\delta(m^2-m_0^2)+m_8^4\sin^2\xi_B\delta(m^2-m_8^2)\}
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
\rho^{(1)}_{08}(m^2) &= \frac{3}{2\sqrt{2}} f_B^{-1} f_Y^{-1}\{m_8^4\sin\xi_B\cos\xi_Y\delta(m^2-m_8^2)-m_0^4\sin\xi_Y\cos\xi_B\delta(m^2-m_0^2)\}\\
&= \rho^{(1)}_{80}(m^2),
\end{aligned}$$

where the sets of indices $\{3, 4, 8, 0\}$ refer to the vector mesons $\{\rho, K^*, \phi, \omega\}$, or the corresponding axial vector mesons $\{A_1, K_A, E, D\}$. The set of mixing angles $\{\xi_Y, \xi_B\}$ designate $\{\theta_Y, \theta_B\}$ for the vector mesons and $\{\psi_Y, \psi_B\}$ for the axial vector mesons. The subscripts Y and B also reference the hypercharge and baryon indices, 8 and 0. Similarly, for the spin 0 mesons,

$$\begin{aligned}
\rho^{(0)}_{11}(m^2) &= \rho^{(0)}_{22}(m^2) = \rho^{(0)}_{33}(m^2) = f_3^2\delta(m^2-m_3^2)\\
\rho^{(0)}_{44}(m^2) &= \rho^{(0)}_{55}(m^2) = \rho^{(0)}_{66}(m^2) = \rho^{(0)}_{77}(m^2) = f_4^2\delta(m^2-m_4^2)\\
\rho^{(0)}_{88}(m^2) &= \tfrac{3}{4} h_Y^2\{\cos^2\chi_Y\delta(m^2-m_8^2)+\sin^2\chi_Y\delta(m^2-m_0^2)\}\\
\rho^{(0)}_{00}(m^2) &= \tfrac{3}{2} h_B^2\{\sin^2\chi_B\delta(m^2-m_8^2)+\cos^2\chi_B\delta(m^2-m_0^2)\}
\end{aligned} \tag{3.2}$$

$$\rho^{(0)}_{08}(m^2) = \frac{3}{2\sqrt{2}} h_B h_Y\{\sin\chi_B\cos\chi_Y\delta(m^2-m_8^2)-\cos\chi_B\sin\chi_Y\delta(m^2-m_0^2)\} = \rho^{(0)}_{80}(m^2),$$

where $\{3, 4, 8, 0\}$ designate the pseudoscalar mesons $\{\pi, K, \eta, \eta'\}$ and scalar mesons $\{\sigma, \kappa, \epsilon, \epsilon'\}$ about which little is known. In what follows, the scalar mesons are omitted. It is also advantageous to introduce the mixing angles $\xi = \theta$ or ψ, and χ obtained from the first and third of equations (2.5) or (2.6) for the nondiagonal, singlet–octet excitations, $\rho_{08}^{(1)}$ and $\rho_{08}^{(0)}$

$$\begin{aligned} \tan \xi &= \frac{m_0}{m_8} \tan \xi_Y = \frac{m_8}{m_0} \tan \xi_B \\ \chi &= \chi_Y = \chi_B. \end{aligned} \tag{3.3}$$

The U(3) sum rules which result from equations (2.5) and (2.6) depend upon choices of A, B, C, D and E as given in equation (2.7) and are summarized in tables 1, 2 and 3 which give coupling constant and (mass)$^{\pm 2}$ sum rules. In tables 1 and 2 coupling constant relationships are given for the quantities $\{f_\rho, f_{K^*}, f_\phi, f_\omega, \theta_Y, \theta_B\}$ and $\{f_{A_1}, f_{K_A}, f_E, f_D, \psi_Y, \psi_B\}$. The corresponding sum rules for the pseudoscalar coupling constants are the same in both q^2 limits, and are the usual unbroken U(3) results,

$$f_\pi^2 = f_K^2 = \tfrac{3}{4} h_Y^2 = \tfrac{3}{2} h_B^2. \tag{3.4}$$

Table 3 lists both (mass)^2 and (mass)$^{-2}$ sum rules resulting from elimination of the coupling constants. As mentioned before, the $q^2 \to 0$ set of sum rules (equations (2.5)) lead to (mass)$^{-2}$ sum rule and the $q^2 \to \infty$ set (equations (2.6)), to (mass)^2 sum rules.

Table 4 gives the mixing angle and isovector masses obtained from consistent solution of the sum rules of table 3. Where two sum rules are given in table 3, m_4, m_8 and m_0 are used to predict ξ and m_3. Where a single sum rule is given, m_4, m_8, m_0 and

Table 1. Coupling constant sum rules for spin 1 mesons in the $q^2 \to 0$ current propagator limit in U(3)

Case	Coupling constant sum rule	Reference
$C = 0$ $D = 0$ $E = 0$	$l_3^{-2} + 2l_4^{-2} = \frac{9}{2} l_B^{-2}$ $4l_4^{-2} - l_3^{-2} = \frac{9}{4} l_Y^{-2}$ $l_3^{-2} - l_4^{-2} = \frac{9}{8} l_B^{-1} l_Y^{-1}(\cos \xi_Y \sin \xi_B - \sin \xi_Y \cos \xi_B)$	Oakes (1968) Wienke and Deshpande (1969)
$D = \frac{1}{8}C$ $E = -\frac{1}{5}\sqrt{2}C$	$3l_3^{-2} + 4l_4^{-2} = 12l_B^{-2} - \frac{3}{4} l_Y^{-2}$ $12l_B^{-2} - 6l_4^{-2} - \frac{3}{2} l_Y^{-2} = \frac{15}{4} l_B^{-1} l_Y^{-1}(\cos \xi_Y \sin \xi_B - \sin \xi_Y \cos \xi_B)$	Chan *et al* (1969)
$C = 0$ $E = 0$	$4l_4^{-2} - l_3^{-2} = \frac{9}{4} l_Y^{-2}$ $2l_4^{-2} - \frac{3}{2} l_Y^{-2} = \frac{3}{4} l_B^{-1} l_Y^{-1}(\cos \xi_Y \sin \xi_B - \sin \xi_Y \cos \xi_B)$	Schwinger (1964)
$C = 0$	$4l_4^{-2} - l_3^{-2} = \frac{9}{4} l_Y^{-2}$	Das *et al* (1967)
$C = 0$ $D = 0$	$4l_4^{-2} - l_3^{-2} = \frac{9}{4} l_Y^{-2}$ $2l_4^{-2} = \frac{3}{4} l_Y^{-2} + \frac{3}{2} l_B^{-2}$	
$E = 0$ $D = 0$	$l_3^{-2} + 2l_4^{-2} = \frac{9}{2} l_B^{-2}$ $l_4^{-2} - \frac{3}{2} l_B^{-2} = \frac{3}{8} l_B^{-1} l_Y^{-1}(\cos \xi_Y \sin \xi_B - \sin \xi_Y \cos \xi_B)$	
$D = 0$	$l_3^{-2} + 2l_4^{-2} = \frac{9}{2} l_B^{-2}$	
$E = 0$	$l_3^{-2} - l_4^{-2} = \frac{9}{8} l_B^{-1} l_Y^{-1}(\cos \xi_Y \sin \xi_B - \sin \xi_Y \cos \xi_B)$	

$\{l\} = \{f_\rho, f_{K^*}, f_\phi, f_\omega\}$ or $\{f_{A_1}, f_{K_A}, f_E, f_D\}$
$\{m\} = \{m_\rho, m_{K^*}, m_\phi, m_\omega\}$ or $\{m_{A_1}, m_{K_A}, m_E, m_D\}$
$\xi = \theta$ or ψ

Table 2. Coupling constant sum rules for spin 1 mesons in the $q^2 \to \infty$ current propagator limit in U(3)

Case	Coupling constant sum rules	Reference
$C=0$ $D=0$ $E=0$	$m_3^4 f_3^{-2}+2m_4^4 l_4^{-2} = \frac{9}{2}(m_8^4\sin^2\xi_B+m_0^4\cos 2\xi_B)l_B^{-2}$ $4m_4^4 l_4^{-2}-m_3^4 l_3^{-2} = \frac{9}{4}(m_8^4\cos^2\xi_Y+m_0^4\sin^2\xi_Y)l_Y^{-2}$ $m_3^4 l_3^{-2}-m_4^4 l_4^{-2} = \frac{9}{8}(m_8^4\cos\xi_Y\sin\xi_B-m_0^4\sin\xi_Y\cos\xi_B)l_B^{-1}l_Y^{-1}$	Oakes (1968) Wienke and Deshpande (1969)
$D=\frac{1}{8}C$ $E=-\frac{1}{5}\sqrt{2}C$	$3m_3^4 l_3^{-2}+4m_4^4 l_4^{-2} = 12(m_8^4\sin^2\xi_B+m_0^4\cos^2\xi_B)l_B^{-2}-\frac{3}{4}(m_8^4\cos\xi_Y+m_0^4\sin^2\xi_Y)l_Y^{-2}$ $12(m_8^4\sin^2\xi_B+m_0^4\cos^2\xi_B)l^{-2}-6m_4^4 l_4^{-2}-\frac{3}{2}(m_8^4\cos^2\xi_Y+m_0^4\sin^2\xi_Y)l_Y^{-2}$ $= \frac{15}{4}(m_8^4\cos\xi_Y\sin\xi_B-m_0^4\sin\xi_Y\cos\xi_B)l_B^{-1}l_Y^{-1}$	Chan *et al* (1969)
$C=0$ $E=0$	$4m_4^4 l_4^{-2}-m_3^4 l_3^{-2} = \frac{9}{4}(m_8^4\cos^2\xi_Y+m_0^4\sin 2\xi_Y)l_Y^{-2}$ $2m_4^4 l_4^{-2}-\frac{3}{2}(m_8^4\cos^2\xi_Y+m_0^4\sin^2\xi_Y)l_Y^{-2} = \frac{3}{4}(m_8^4\cos\xi_Y\sin\xi_B-m_0^4\sin\xi_Y\cos\xi_B)l_B^{-1}l_Y^{-1}$	Schwinger (1964)
$C=0$	$4m_4^4 l_4^{-2}-m_3^4 l_3^{-2} = \frac{9}{4}(m_8^4\cos^2\xi_Y+m_0^4\sin^2\xi_Y)l_Y^{-2}$	Das *et al* (1967)
$C=0$ $D=0$	$4m_4^4 l_4^{-2}-m_3^4 l_3^{-2} = \frac{9}{4}(m_8^4\cos^2\xi_Y+m_0^4\sin^2\xi_Y)l_Y^{-2}$ $2m_4^4 l_4^{-2} = \frac{3}{2}(m_8^4\sin^2\xi_B+m_0^4\cos^2\xi_B)l_B^{-2}+\frac{3}{4}(m_8^4\cos^2\xi_Y+m_0^4\sin^2\xi_Y)l_Y^{-2}$	
$E=0$ $D=0$	$m_3^4 l_3^{-2}+2m_4^4 l_4^{-2} = \frac{9}{2}(m_8^4\sin^2\xi_B+m_0^4\cos^2\xi_B)l_B^{-2}$ $m_4^4 l_4^{-2}-\frac{3}{4}(m_8^4\sin^2\xi_B+m_0^4\cos^2\xi_B) = \frac{3}{8}(m_8^4\cos\xi_Y\sin\xi_B-m_0^8\sin\xi_Y\cos\xi_B)l_B^{-1}l_Y^{-1}$	
$D=0$	$m_3^4 l_3^{-2}+2m_4^4 l_4^{-2} = \frac{9}{2}(m_8^4\sin^2\xi_B+m_0^4\cos^2\xi_B)l_B^{-2}$	
$E=0$	$m_3^4 l_3^{-2}-m_4^4 l_4^{-2} = \frac{9}{8}(m_8^4\cos\xi_Y\sin\xi_B-m_0^4\sin\xi_Y\cos\xi_B)l_B^{-1}l_Y^{-1}$	

$\{l\} = \{f_\rho, f_{K^*}, f_\phi, f_\omega\}$ or $\{f_{A_1}, f_{K_A}, f_E, f_D\}$
$\{m\} = \{m_\rho, m_{K^*}, m_\phi, m_\omega\}$ or $\{m_{A_1}, m_{K_A}, m_E, m_D\}$
$\xi = \theta$ or ψ

Table 3. Mass sum rules for spin 1 and spin 0 mesons in the $q^2 \to 0$ and $q^2 \to \infty$ current propagator limit in U(3)

Case	Mass sum rule	Reference
$C=0$ $D=0$ $E=0$	$m_3^{\pm 2} = m_0^{\pm 2}$ $2m_4^{\pm 2} = m_8^{\pm 2}+m_0^{\pm 2}$ $\tan 2\xi = 2\sqrt{2}$	Oakes (1968) Wienke and Deshpande (1969)
$D=\frac{1}{8}C$ $E=-\frac{1}{5}\sqrt{2}C$	$3m_3^{\pm 2}+4m_4^{\pm 2} = 8(m_8^{\pm 2}\sin^2\xi+m_0^{\pm 2}\cos^2\xi)-(m_8^{\pm 2}\cos^2\xi+m_0^{\pm 2}\sin^2\xi)$ $8(m_8^{\pm 2}\sin^2\xi+m_0^{\pm 2}\cos^2\xi)-6m_4^{\pm 2}-2(m_8^{\pm 2}\cos^2\xi+m_0^{\pm 2}\sin^2\xi)$ $= \frac{5}{4}\sqrt{2}(m_0^{\pm 2}-m_8^{\pm 2})\sin 2\xi$	Chan *et al* (1969)
$C=0$ $E=0$	$4m_4^{\pm 2}-m_3^{\pm 2} = 3(m_8^{\pm 2}\cos^2\xi+m_0^{\pm 2}\sin^2\xi)$ $2m_4^{\pm 2}-2(m_8^{\pm 2}\cos^2\xi+m_0^{\pm 2}\sin^2\xi) = \frac{1}{4}\sqrt{2}(m_0^{\pm 2}-m_8^{\pm 2})\sin 2\xi$	Schwinger (1964)
$C=0$	$4m_4^{\pm 2}-m_3^{\pm 2} = 3(m_8^{\pm 2}\cos^2\xi+m_0^{\pm 2}\sin^2\xi)$	Das *et al* (1967)
$C=0$ $D=0$	$4m_4^{\pm 2}-m_3^{\pm 2} = 3(m_8^{\pm 2}\cos^2\xi+m_0^{\pm 2}\sin^2\xi)$ $2m_4^{\pm 2} = m_8^{\pm 2}+m_0^{\pm 2}$	
$E=0$ $D=0$	$m_3^{\pm 2}+2m_4^{\pm 2} = 3(m_8^{\pm 2}\sin^2\xi+m_0^{\pm 2}\cos^2\xi)$ $m_4^{\pm 2}-(m_8^{\pm 2}\sin^2\xi+m_0^{\pm 2}\cos^2\xi) = \frac{1}{8}\sqrt{2}(m_0^{\pm 2}-m_8^{\pm 2})\sin 2\xi$	
$D=0$	$m_3^{\pm 2}+2m_4^{\pm 2} = 3(m_8^{\pm 2}\sin^2\xi+m_0^{\pm 2}\cos^2\xi)$	
$E=0$	$m_3^{\pm 2}-m_4^{\pm 2} = \frac{3}{8}\sqrt{2}(m_0^{\pm 2}-m_8^{\pm 2})\sin 2\xi$	

$\{m\} = \{m_\rho, m_{K^*}, m_\phi, m_\omega\}$ or $\{m_{A_1}, m_{K_A}, m_E, m_D\}$ or $\{m_\pi, m_K, m_\eta, m_{\eta'}\}$
$\xi = \theta$ or ψ or χ

Table 4. Sum rule predictions of mixing angles and isovector masses in U(3)

	Vector				Axial vector				Pseudoscalar				
	(mass)^2		(mass)$^{-2}$		(mass)^2		(mass)$^{-2}$		(mass)^2		(mass)$^{-2}$		
Case	θ	m_ρ	θ	m_ρ	ψ	m_{A_1}	ψ	m_{A_1}	χ	m_π	χ	m_π	Reference
$C = 0$, $D = 0$, $E = 0$	35·3	780	35·3	780	35·3	1285	35·3	1285	35·3	958	35·3	958	Oakes (1968) Wienke and Deshpande (1969)
$D = \frac{1}{8}C$, $E = -\frac{1}{5}\sqrt{2}C$	32·8	778	37·4	783	6·7	1314	NS		56·4	860	87·3	578	Chan *et al* (1969)
$C = 0$, $E = 0$	39·3	760	32·3	793	NS		NS		NS		NS		Schwinger (1964)
$C = 0$	39·9		28·8		NS		67·9		9·3		NS		Das *et al* (1967)
$C = 0$, $D = 0$	35·8		29·6		NS		NS		64·8	1202	NS		
$E = 0$, $D = 0$	31·5	771	38·7	785	NS		NS		NS		NS		
$D = 0$	31·3		36·5		NS		NS		NS		NS		
$E = 0$	32·5		NS		NS		NS		21·6		NS		

NS: no solution

m_3 are used to determine ξ. The case $C = D = E$ results in an independent determination of the mixing angle such that

$$\tan 2\xi = 2\sqrt{2} \tag{3.5}$$

and

$$m_3 = m_0. \tag{3.6}$$

From table 4 it is obvious that the U(3) breakings examined adequately describe the vector mesons, but are inadequate for the axial vector and pseudoscalar mesons. Dropping the requirement of consistency of the sets of sum rules would allow determination of the mixing angle (or m_3, m_4, m_8, m_0, given ξ) from one or another equations. The question of which equation, can be dictated by consideration of SU(3) symmetry for the lowest order spectral moment integrals.

If in equations (2.5) and (2.6) we split off the unitary singlet contributions to the spectral integrals by writing

$$\begin{aligned} s^{(1)}\delta_{ab} &\rightarrow s^{(1)}\delta_{ab} + s^{(1)\prime}\delta_{a0}\delta_{b0} \\ s^{(0)}\delta_{ab} &\rightarrow s^{(0)}\delta_{ab} + s^{(0)\prime}\delta_{a0}\delta_{b0}, \end{aligned} \tag{3.7}$$

we suppress the contributions of the spectral functions $\rho_{00}^{(1)}$ and $\rho_{00}^{(0)}$ in the calculation and obtain sum rules in SU(3). The effect of $s^{(i)\prime}$ is to scale the singlet contribution differently from the octet contributing pieces. In equation (3.4), the term $\frac{3}{2}h_B^2$ does not appear and the effects on the sum rules given in tables 1, 2 and 3 are easily explained rather than retabulated. In table 1, where two or more sum rules appear for particular choice of C, D or E, the prescription is to eliminate l_B^{-2} from the equations, reducing the number of sum rules by one. If this is not possible, the sum rule with l_B^{-2} is suppressed. In table 2, the identical procedure is applied to the combination $(m_8^4 \sin^2 \xi_B + m_0^4 \cos^2 \xi_B) l_B^{-2}$ and in table 3 to the combination $(m_8^{\pm 2} \sin^2 \xi + m_0^{\pm 2} \cos^2 \xi)$. In addition in table 3 the two conditions $m_3^{\pm 2} = m_0^{\pm 2}$, $\tan 2\xi = 2\sqrt{2}$ no longer hold and the bare combination $(m_8^{\pm 2} + m_0^{\pm 2})$ is replaced by $\frac{1}{2}\{m_3^{\pm 2} + 3(m_8^{\pm 2}\cos^2 \xi + m_0^{\pm 2} \sin^2 \xi)\}$. A number of the resulting sum rules are well known for SU(3) (Gell-Mann 1962, 1964, Oakes and Sakurai 1967, Das *et al* 1967, Oakes 1968, Wienke and Dashpande 1969).

In table 5, predictions of isovector masses and mixing angles, following solution of the SU(3) mass sum rules as modified above for table 3, are given. The procedures yielding table 4 are again followed in compiling table 5. It is simple to see in table 5 that the cases reported result from a single sum rule which is used as a prediction of the mixing angle except the particular case $C = 0$, $E = 0$ which is identical to the corresponding case in table 4 for U(3) symmetry. The set of mixing angles, (39·9, NS, 9·3) for (mass)^2 and (28·8, 67·9, NS) for (mass)$^{-2}$ sum rules, result from the well known Gell-Mann–Okubo mass formula (Gell-Mann 1961, Okubo 1962) in its (mass)$^{\pm 2}$ form

$$4m_4^{\pm 2} - m_3^{\pm 2} = 3(m_8^{\pm 2}\cos^2 \xi + m_0^{\pm 2}\sin^2 \xi) \tag{3.8}$$

which is traditionally used to define the SU(3) mixing angles in the (mass)^2 case for the meson octets. The mixing angle and isovector mass predictions listed for the vector, axial vector and pseudoscalar mesons in tables 4 and 5 are to be compared with their experimental values:

$$\begin{aligned} m_\rho &= 765\ \text{MeV}, & \theta &= 39{\cdot}5^\circ\text{–}22{\cdot}4^\circ \\ m_{A_1} &= 1070\ \text{MeV}, & \psi &= \text{unknown} \\ m_\pi &= 140\ \text{MeV}, & \chi &= 10{\cdot}4^\circ, \end{aligned}$$

Table 5. Sum rule predictions of mixing angles and isovector masses in SU(3)

	Vector				Axial vector				Pseudoscalar				
	(mass)^2		(mass)$^{-2}$		(mass)^2		(mass)$^{-2}$		(mass)^2		(mass)$^{-2}$		
Case	θ	m_ρ	θ	m_ρ	ψ	m_{A_1}	ψ	m_{A_1}	χ	m_π	χ	m_π	Reference
$C = 0$ $D = 0$ $E = 0$	39·9		28·8		NS		67·9		9·3		NS		Oakes (1967) Wienke and Deshpande (1969)
$D = \frac{1}{8}C$ $E = -\frac{1}{5}\sqrt{2C}$	34·2		47·7		NS		NS		NS		NS		Chan *et al* (1969)
$C = 0$ $E = 0$	39·3	760	32·3	793	NS		NS		NS		NS		Schwinger (1964)
$C = 0$	39·9		28·8		NS		67·9		9·3		NS		Das *et al* (1967)
$C = 0$ $D = 0$	39·9		28·8		NS		67·9		9·3		NS		
$E = 0$ $D = 0$	32·5		NS		NS		NS		21·6		NS		
$E = 0$	32·5		NS		NS		NS		21·6		NS		

NS: no solution

given by the Particle Data Group (1971). Simultaneous prediction of the isovector mass and mixing angle is precluded for the pseudoscalar and axial vector mesons, as seen in tables 4 and 5, but not for the vector mesons. Since the experimental estimates of m_ρ and θ vary within the range given in tables 4 and 5, the sum rules given seem appropriate for the vector mesons. Further experimental refinement is necessary, however, to distinguish between the different symmetry breaking schemes listed.

We have concentrated this analysis on the 0^-, 1^+, and 1^- octets and nonets of mesons. Similar analysis for higher order J^P candidates might prove interesting and the hope is to report these findings later.

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Appendix

A brief discussion of symmetry breaking, mass and current mixing as applied in this analysis is given for completeness.

The most general isospin conserving, symmetric, unitary symmetry breaking interaction can be written in the form

$$t_{ab} = A\delta_{ab} + Bd_{8ab} + C\delta_{a8}\delta_{b8} + D\delta_{a0}\delta_{b0} + E(\delta_{a0}\delta_{b8} + \delta_{a8}\delta_{b0}) \tag{A.1}$$

corresponding to a nonet of non-zero mass particles ($A \neq 0$) split into four isospin multiplets. To obtain t_{ab} for singlet$\oplus$octet$\oplus$twenty-sevenplet breaking we take octet matrix elements of an operator k, containing pieces that transform as a singlet, octet and twenty-sevenplet and equate them to the right hand side of equation (A.1):

$$t_{ab} = \langle a|k^1|b\rangle + \langle a|k^8|b\rangle + \langle a|k^{27}|b\rangle \tag{A.2}$$

and find after obtaining Clebsch–Gordan coefficients

$$\begin{aligned}
t_{11} &= t_{22} = t_{33} = A + B/\sqrt{3} = -\sqrt{\tfrac{1}{8}}\langle k^1\rangle + \sqrt{\tfrac{1}{5}}\langle k^8\rangle + \sqrt{\tfrac{1}{120}}\langle k^{27}\rangle \\
t_{44} &= t_{55} = t_{66} = t_{77} = A - B/\sqrt{12} = -\sqrt{\tfrac{1}{8}}\langle k^1\rangle - \sqrt{\tfrac{1}{20}}\langle k^8\rangle - \sqrt{\tfrac{3}{40}}\langle k^{27}\rangle \\
t_{88} &= A - B/\sqrt{3} + C = -\sqrt{\tfrac{1}{8}}\langle k^1\rangle - \sqrt{\tfrac{1}{5}}\langle k^8\rangle + \sqrt{\tfrac{27}{40}}\langle k^{27}\rangle \\
t_{00} &= A + D = -\sqrt{\tfrac{1}{8}}\langle k^1\rangle \\
t_{08} &= t_{80} = \sqrt{\tfrac{2}{3}}B + E = \sqrt{\tfrac{2}{5}}\langle k^8\rangle
\end{aligned} \tag{A.3}$$

with $\langle k^1\rangle$, $\langle k^8\rangle$ and $\langle k^{27}\rangle$ the reduced matrix elements of the singlet, octet and twenty-sevenplet transforming pieces. Obviously, for pure singlet$\oplus$octet splitting, $\langle k^{27}\rangle = 0$

$$\begin{aligned}
A &= -\sqrt{\tfrac{1}{8}}\langle k^1\rangle \\
B &= \sqrt{\tfrac{3}{5}}\langle k^8\rangle \\
C &= D = E = 0
\end{aligned} \tag{A.4}$$

and for singlet$\oplus$octet$\oplus$twenty-sevenplet breaking,

$$\begin{aligned} A &= -\sqrt{\tfrac{1}{8}}\langle k^1\rangle - \tfrac{5}{3}\sqrt{\tfrac{1}{120}}\langle k^{27}\rangle \\ B &= \sqrt{\tfrac{3}{5}}\langle k^8\rangle + \tfrac{8}{3}\sqrt{\tfrac{3}{120}}\langle k^{27}\rangle \\ C &= \tfrac{40}{3}\sqrt{\tfrac{1}{120}}\langle k^{27}\rangle \\ D &= \tfrac{1}{8}C \\ E &= -\tfrac{1}{5}\sqrt{2}C. \end{aligned} \tag{A.5}$$

The absolute values of $\langle k^1\rangle$, $\langle k^8\rangle$, $\langle k^{27}\rangle$, (or A, B, C, D, E) are unimportant since those constants are eliminated in obtaining the sum rules.

The choice $C = E = 0$ gives the above octet breaking, without mixing, while splitting off the singlet piece in the nonet U(3) scheme (Schwinger 1964), while $C = 0$ allows for mixing (Das *et al* 1967). Other choices for the parameters as listed in tables 1–4 are given for completeness and do not correspond to any generally employed symmetry breaking mechanisms.

The respective terms mass mixing and current mixing (Coleman and Schnitzer 1964, Kroll *et al* 1967) refer to the symmetry properties of the mass and kinetic terms in some assumed lagrangian describing the spin 0 and spin 1 meson and their couplings. If the kinetic terms transform as some representation of assumed symmetry group for the lagrangian, we speak of current (or vector) mixing. If the mass terms, instead, transform as some representation, we refer to this as mass (or particle) mixing. Mass mixing is a simple and reasonable starting point for the spin 0 mesons that can also be described within the framework of a Schrödinger equation acting on the space of one-particle states. Within the framework of the pole approximations, mass mixing has been a suitable approximation for describing a large class of interactions of spinless mesons. However, for vector particle interactions mass mixing is inconsistent. Among a number of other important considerations, mass mixing for the vector mesons destroys the U(3) commutation relationships of equations (1.1) and the symmetry of the c number Schwinger terms assumed for this analysis. Furthermore, current mixing has been shown (Oakes and Sakurai 1967) to be the only theory of ω–ϕ mixing compatible with the sum rules of Weinberg (equation (1.3)).

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